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Calculation of Propeller-Excited Whirling Critical Speeds

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This paper presents a calculation method, suitable for manual calculation or use with a small, desk-top calculator, for estimating the propeller-excited whirling critical speed of a shafting system. The method is especially adaptable to use with a small programmable calculator. Based on a two-support model of the propeller and tailshaft similar to that used in earlier whirling calculations, this method "brackets" the natural frequency by first assuming the forward end of the tailshaft to be simply supported, then repeating the calculation, assuming the forward end to be fixed. This gives an upper and lower limit on the natural frequency. The actual natural frequency is estimated by interpolation between these two values. The proposed calculation method includes propeller gyroscopic and inertia effects, as well as shaft mass effects. Entrained water may also be taken into account. Comparisons between the results of the proposed hand calculation and the results of other well-known methods of whirling analysis are presented for a typical vessel shafting system. Extensions of the proposed calculation to include the effects of the line shafting and the sterntube bearing stiffness are also presented.

Introduction

THE PHENOMENON of whirling vibration of propulsion shafting has received considerable attention in recent years. However, virtually all of the recently developed analytical methods for predicting whirling critical speeds have been computer-oriented. As a result, workable manual calculation procedures of comparable accuracy have not been developed, even though the theoretical basis for such a procedure [1][2] has existed for some time. It is this deficiency which we presently hope to remedy.

We will assume that the unbalance-excited critical speed of the shaft lies well above the maximum shaft speed, as is typical of present-day merchant ships. We may then concentrate on determining the propeller-excited critical speed, which usually occurs when the product of the shaft rpm and the number of propeller blades equals the whirling natural frequency of the shaft system.

This brings out an interesting feature of this particular type of vibration; namely, that at the critical speed the shaft will whirl much faster than it rotates. The basic theory governing this situation is given in reference [1] and will not be repeated here.

It should be noted, however, that such a system will have two distinct natural frequencies, one in which the directions of whirl and rotation are the same (forward whirl) and one in which they are opposite (reverse whirl).

For an insight into the historical background of this type of calculation, the reader is referred to [2] and [3], and especially to the discussion of [2] by Lewis. It will be noted that the calculation procedure used here is substantially the same as that followed by Lewis, except that a number of important improvements have been made to simplify the calculations and increase the accuracy of the results. The most important of these improvements is the ability to "bracket" the actual natural frequency. Previous hand calculations have generally been biased to given either a lower or upper frequency limit. We will obtain both, and interpolate between the two to obtain the best possible frequency estimate, for both forward and reverse whirl.

Development of calculation procedure

Initially, we will take as our model the shaft system shown in Fig. 1. The shaft will be assumed to be rotating at a known rate of Ω radians/second (rad/s). The (unknown) whirling natural frequency will be ω rad/s.

The N-bladed propeller will have a weight of W and a moment of inertia I_d about its diameter. (Note that I_d is usually taken as half the polar moment of inertia of the propeller.)

The shaft will be of uniform diameter, and will have a second moment of area (in bending) I, a modulus of elasticity E, and a mass per unit length μ.

The bearings will be considered infinitely stiff. The after bearing will be considered a simple support at all times. The forward bearing will be considered first as a simple support, then as a clamped-end support.

The dimension b will be taken from the center of gravity of the propeller to a point within the after bearing, one shaft diameter inboard from the bearing aft end.

The dimension l is then taken from this point forward to the center of the forward bearing.

The diameter of the shaft will be taken as the nominal tailshaft diameter between journals. Increases in diameter at the journals may be safely ignored, as may the taper of the tailshaft inside the propeller hub.

Any consistent set of units may be used. Typical units are millimetres or inches for length, kilograms or pounds for weight or force, kg-mm-s^2 or lb-in.-s^2 for I_d, and kg-s^2/mm^2 or lb-s^2/in.^{2} for μ. Units for the remaining quantities are easily derived from the basic units.

For this system, we first determine four influence coefficients, defined as follows:

α_{11} = deflection at propeller due to a unit force applied at the propeller

α_{12} = angular deflection at propeller in radians (for small angles, we may use the slope at propeller, mm/mm or in./in.) due to a unit force applied at the propeller

α_{21} = deflection at propeller due to a unit moment applied at the propeller (by Maxwell's reciprocity relationship, $\alpha_{12} = \alpha_{21}$)

α_{22} = angular deflection at propeller in radians (or, for small angles, slope at propeller) due to a unit moment applied at the propeller

These coefficients may be determined by double integration of the shaft bending moment equation, for both a simply supported shaft and a shaft having the forward end clamped. The results are as follows:

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[2] Numbers in brackets designate References at end of paper.

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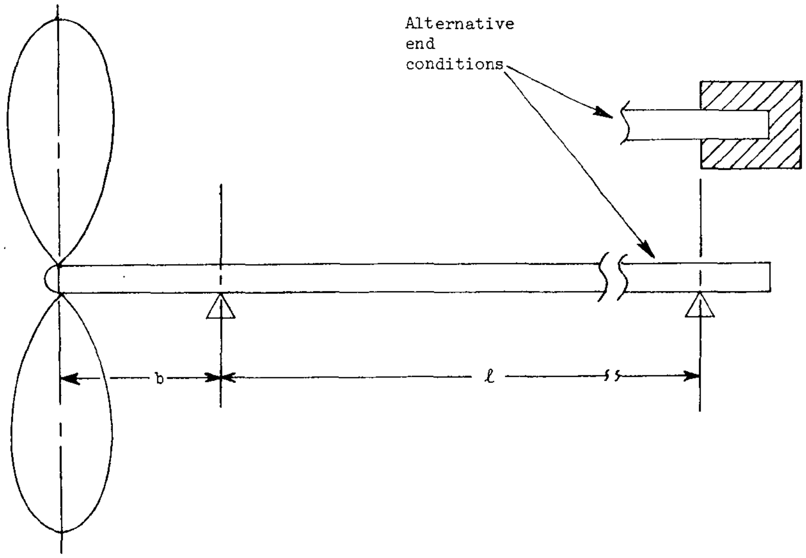

Fig. 1 Simplified model of tailshaft

	Simple Support	Fixed Forward End
α_{11}	$\frac{b^3}{3EI} + \frac{b^2 l}{3EI}$	$\frac{b^3}{3EI} + \frac{b^2 l}{4EI}$
$\alpha_{12}\ (= \alpha_{21})$	$\frac{b^2}{2EI} + \frac{bl}{3EI}$	$\frac{b^2}{2EI} + \frac{bl}{4EI}$
α_{22}	$\frac{b}{EI} + \frac{l}{3EI}$	$\frac{b}{EI} + \frac{l}{4EI}$

Before proceeding further, we note the following characteristics of these influence numbers:

1. In each case, the first term depends only on b, regardless of the length or fixity of the forward part of the shaft, and is a measure of the flexibility of the shaft overhang.
2. After finding any one influence coefficient, the remaining coefficients may be found with a minimum of effort by taking advantage of the similarities in form among the various terms.

Based on the preceding influence coefficients, Den Hartog [1] derives and presents the following equation for the whirling natural frequency of a disk on a rotating shaft, *neglecting the mass of the shafting:*

$$F^4 - 2SF^3 + \left(\frac{D+1}{D(E-1)}\right)F^2 - 2\left(\frac{S}{E-1}\right)F - \frac{1}{D(E-1)} = 0$$

where

$$F = \omega\sqrt{\alpha_{11} m}$$

$$D = \frac{I_d \alpha_{22}}{m \alpha_{11}}$$

$$E = \frac{{\alpha_{12}}^2}{\alpha_{11}\alpha_{22}}$$

$$S = \Omega\sqrt{\alpha_{11} m}$$

Note that the foregoing groups of variables are dimensionless.

Of the quantities just given, the only one not previously defined is m, the propeller mass. To simplify later calculations, we may add to the propeller mass one-third the mass of the overhanging shaft, thereby making the reasonable assumption (which may be checked by examining the influence numbers) that flexure of the shaft overhang will be small compared with the overall propeller deflection. Corrections for the mass of the entrained water may also be made at this point, along with corrections for the effect of entrained water on the propeller inertia I_d. As a rough approximation, both the mass and the inertia may be increased by 25 percent to account for this effect. The equation also includes a consideration of the propeller gyroscopic effect, without any additional effort on our part.

Since we are looking for a propeller-excited critical speed, in which the excitation forces (and the resulting whirl) occur at blade rate, we may set

$$\omega = \pm N\Omega$$

That is to say, the whirling natural frequency ω will be N times the shaft rotational speed Ω, and may be in the same direction (forward whirl) or in the opposite direction (reverse whirl).

Returning to our previously defined dimensionless groups, we then have

$$F = \pm NS$$

or

$$S = \pm\frac{F}{N}$$

Substituting this relation back into the original fourth-degree frequency equation and simplifying, we obtain the following:

$$\left(1 \mp \frac{2}{N}\right)(D)(E-1)F^4 + \left[\left(1 \mp \frac{2}{N}\right)(D) + 1\right]F^2 - 1 = 0$$

This is a quadratic in F^2, and may be solved by the quadratic formula. After simplifying, we obtain our first working expression, which is very much similar to Jasper's formula No. 1 [3]

$$\omega_1^2 = \frac{B \mp \sqrt{B^2 - 4A}}{2\alpha_{11} m A}$$

where

$$A = \left(1 \mp \frac{2}{N}\right)(D)(1.0 - E)$$

$$B = \left(1 \mp \frac{2}{N}\right)(D) + 1.0$$

(use minus sign for forward whirl, plus sign for reverse whirl)

Note the subscript used with ω, which indicates that this is, in fact, only a partial solution to the overall problem.

The remaining quantities are as previously defined.

Now that we have the natural frequency of our model system, neglecting only the mass of the shafting between the bearings, we may refer to any standard text on vibrations to help complete the calculations. First, we apply standard formulas to determine the (lateral) natural frequency of the *shafting alone,* neglecting the rotational inertia of the shaft, but including the mass of the shaft. We have

$$\omega_2{}^2 = \frac{KEI}{\mu l^4}$$

where

K = 97.4 for a simply supported shaft
= 237.2 for a shaft with clamped forward end
μ = shaft mass per unit length, $\text{lb-s}^2/\text{in.}^2$ or $\text{kg-s}^2/\text{mm}^2$

E, I and l are as previously defined.

Note that ω is subscripted here as well. This relation is independent of the direction of whirl.

Finally, we apply Dunkerley's equation to obtain the combined system natural frequency from these two partial system natural frequencies. Thus

$$\frac{1}{\omega^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2}$$

This completes development of the basic calculations. Before proceeding further, it should be noted that a number of such calculations are necessary to fully describe a shafting system. Thus

1. The value ω_1^2 should be calculated for each of the following cases:
   (*a*) simple support, forward whirl
   (*b*) simple support, reverse whirl
   (*c*) fixed forward end, forward whirl
   (*d*) fixed forward end, reverse whirl
2. The value ω_2^2 should be calculated for both of these cases:
   (*a*) simple support
   (*b*) fixed forward end

Note once again that the ω_2^2 values are independent of the direction of whirl.

3. The combined system natural frequency should be obtained by Dunkerley's equation for each of the four cases listed in Calculation 1. Be careful to use the ω_2^2 value which was calculated for the same end conditions as the ω_1^2 value in each case.
4. We will then have the following four natural frequency values for the overall system:
   (*a*) simple support, forward whirl
   (*b*) simple support, reverse whirl
   (*c*) fixed forward end, forward whirl
   (*d*) fixed forward end, reverse whirl

Note that the calculated values are in radians per second. We may now convert them and find the critical rpm's:

$$\text{Vibrations per minute} = 9.55 \times \omega$$

$$\text{Critical shaft rpm} = \frac{\text{Vibrations per minute}}{N}$$

Note also that N may be replaced throughout the calculation by $2N$ to investigate a problem involving twice-blade-rate excitation, or by any other multiple of N as required.

Our four natural frequency values may best be used to bracket the actual natural frequencies. The actual natural frequency for either forward or reverse whirl may be expected to lie between that obtained for a simply supported shaft and that obtained for a shaft having the forward end fixed. Lacking an estimate of the actual fixity of the forward end, the arithmetic mean of the two values may be used to estimate the actual natural frequency with a fair degree of accuracy.

Approximately two hours' total work with an ordinary pocket calculator is required to obtain this complete description of the shafting system whirling behavior, which should be comparable in accuracy to that obtained from a digital computer program.

For a quicker estimate, it is often sufficient to determine only an upper or lower limit for whirling. This may be done as follows:

(*a*) For an *upper* limit on the critical speed, perform calculations for a *shaft with fixed end* in *forward whirl* only.

(*b*) For a *lower* limit on the critical speed, perform calculations for a *simply supported shaft* in *reverse whirl* only.

This simplification should reduce the time required for the calculation to one hour or less with an ordinary pocket calculator. Further reductions in calculation time may be achieved by rearranging the computations for more efficient use of the calculator.

It is possible also to write a program for performing these calculations on a programmable calculator. Such a program, consisting of 295 steps, has been written for the Olivetti P652 calculator currently in use at the American Bureau of Shipping (ABS). For use on smaller-capacity machines, it is possible to break down the overall program into stages. For example, the calculations for determining the six influence coefficients require approximately 90 steps, while the calculation of any one of the four values of ω_1 is accomplished in a subroutine consisting of approximately 50 steps. Of course, programs of this length are best suited to calculators equipped with magnetic card program storage.

A summary of the calculations described in the foregoing is included as Appendix 1.

Consideration of line shafting

Using the formula previously given for calculating the natural frequency of the shaft alone, we may also determine the natural frequency of each separate span of line shafting, assuming simple support at the bearings. We neglect the effects of any shafting forward or aft of the span under consideration. However, where the forwardmost shaft span is connected by an integral flange to the reduction gear, we may consider this end to be clamped. We may then make the following observations:

1. In order to avoid participation of the line shafting in propeller-excited whirling vibrations, the natural frequency of each span should be well above the natural frequency of the propeller/tailshaft combination. This principle may be employed in determining the spacing of the line shaft bearings. Note that the assumption of a fixed forward end for the shaft span adjacent to the reduction gear will permit this span to be somewhat longer than would otherwise be the case. This provision will help reduce the likelihood of alignment difficulties.

2. The natural frequency of the shaft span just ahead of the tailshaft may be used to help assess the fixity of the forward end of the tailshaft, thereby refining the original natural frequency estimates for the tailshaft/propeller combination. If the line shaft and the tailshaft/propeller natural frequencies are very close, then the fixity of the tailshaft forward end will be lower than 50 percent. If the line shaft natural frequency is much higher than the tailshaft/propeller natural frequency, then the fixity of the tailshaft forward end may be more than 50 percent. The actual numerical relation of these quantities remains to be established. However, the following two limiting conditions will aid in this assessment: (i) If the line shaft natural frequency is exactly equal to that of the tailshaft/propeller combination, then the two shafts will vibrate together at the critical speed. The line shaft will therefore exert little or no end moment on the tailshaft, so that the fixity of the tailshaft forward end will be virtually zero. (ii) If the line shaft natural frequency is extremely high, perhaps several orders of magnitude higher than that of the tailshaft/propeller combination, then the line shaft will be found to be very stiff compared with the tailshaft. The line shaft will therefore exert a very large end moment on the tailshaft, so that the fixity of the tailshaft forward end will approach 100 percent.

Conclusions and additional considerations

It should be noted that our calculation procedure is general in nature, and has approximately the same scope of applicability as the procedures given in references [2–4]. This permits meaningful cross-checking of the results of the various methods.

The results obtained by the method developed in the preceding have been compared with the results obtained by Panagopulos's formula and those obtained by applying the ABS computer program PROPULSOR [4] for a number of vessels. In each case, the results obtained by our proposed method have been in close agreement with those obtained by computer. It should be noted that the basic theory and underlying assumptions used in the PROPULSOR program are very similar to those used herein, although the PROPULSOR program was independently developed and is capable of considering numerous other factors which are beyond the scope of a hand calculation.

A summary of the results of this comparison for a typical shaft line is given in Appendix 2.

A proposed calculation procedure to account for the stiffness of the propeller-end bearing is presented in Appendix 3.

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References

1 Den Hartog, J. P., *Mechanical Vibrations*, 4th ed., McGraw-Hill, New York, 1956.

2 Panagopulos, E., "Design Stage Calculations of Torsional, Axial, and Lateral Vibrations of Marine Shafting," *Trans.* SNAME, Vol. 58, 1950.

3 Jasper, N. H. "A Design Approach to the Problem of Critical Whirling Speeds of Shaft-Disk Systems," DTMB Report 890, David Taylor Model Basin, Dec. 1954.

4 Salzman, R. H. and Kauneckas, W. J., "PROPULSOR Computer Program," American Bureau of Shipping, New York.

Appendix 1

Summary of working formulas

Propeller on weightless shaft

$$\omega_1^2 = \frac{B \mp \sqrt{B^2 - 4A}}{2\alpha_{11} m A}$$ (note: in general, only the − sign will be used)

ω = natural frequency, rad/s

$$\left.\begin{aligned} A &= \left(1 \mp \frac{2}{N}\right)(D)(1.0 - E) \\ B &= \left(1 \mp \frac{2}{N}\right)(D) + 1.0 \end{aligned}\right\} \text{use} \begin{array}{l} - \text{ sign for forward whirl} \\ + \text{ sign for reverse whirl} \end{array}$$

N = number of propeller blades

$D = I_D \alpha_{22}/m\alpha_{11}$

$E = (\alpha_{12})^2/\alpha_{11}\alpha_{22}$

I_d = propeller diametral moment of inertia (= 1/2 polar moment of inertia) (corrected for entrained water), lb-in.-s^2 or kg-mm-s^2

m = mass of propeller + 1/3 mass of overhanging shaft, lb-s^2/in. or kg-s^2/mm (corrected for entrained water)

α_{11}, α_{12} and α_{22} are influence numbers, defined as follows:

α_{11} = deflection at propeller, in., (or mm), due to a 1-lb (or 1-kg) force at propeller

α_{12} = slope at propeller, in./in., (or mm/mm), due to a 1-lb (or 1-kg) force at propeller
= deflection at propeller, in., (or mm), due to a 1-in.-lb (or 1 kg-mm) moment at propeller

α_{22} = slope at propeller, in./in., (or mm/mm), due to a 1-in.-lb (or 1 kg-mm) moment at propeller

A table of these influence numbers is:

Coefficient	Simple Support	Fixed Forward End
α_{11}	$\frac{b^3}{3EI} + \frac{b^2 l}{3EI}$	$\frac{b^3}{3EI} + \frac{b^2 l}{4EI}$
α_{12}	$\frac{b^2}{2EI} + \frac{bl}{3EI}$	$\frac{b^2}{2EI} + \frac{bl}{4EI}$
α_{22}	$\frac{b}{EI} + \frac{l}{3EI}$	$\frac{b}{EI} + \frac{l}{4EI}$

where

b = length of shaft overhang, in. or mm
l = length of shaft between bearings, in. or mm
E = Young's modulus of shaft material, psi or kg/mm^2
I = second moment of area of shaft in bending, in.^{4} or mm^4

Natural frequency of shaft alone (span between bearings *ONLY*)

(This formula may also be used to find the natural frequency of line shaft span.):

$$\omega_2^2 = \frac{KEI}{Ml^4}$$

where

M = shaft mass per unit length, lb.-s.^{2}/in.^{2} or kg-s^2/mm^2
K = 97.4 for simply supported shaft
= 237.2 for shaft having fixed forward end

Combined natural frequency

$$\omega = \sqrt{\frac{1}{1/\omega_1^2 + 1/\omega_2^2}} = \sqrt{\frac{\omega_1^2\,\omega_2^2}{\omega_1^2 + \omega_2^2}}$$

ω_1 = natural frequency of propeller on weightless shaft
ω_2 = natural frequency of shaft alone
ω = overall system natural frequency (shaft end conditions of ω_1 and ω_2 must be the same for each ω)

NOTE: *In general, four calculations of overall natural frequency are required, namely:*

1. forward whirl, simple support
2. forward whirl, fixed forward end
3. reverse whirl, simple support
4. reverse whirl, fixed forward end

The natural frequencies obtained for a fixed forward end and for simple support may be averaged to obtain a single frequency estimate for each direction of whirl.

Appendix 2

Comparison of results of frequency calculations

The shaft line model for which our comparison checks will be performed is shown in Fig. 2. The whirling critical speeds of this shaft system will be calculated by the method developed herein, and checked against the results obtained by Panagopulos's formula [2] and by the ABS computer program PROPULSOR [4].

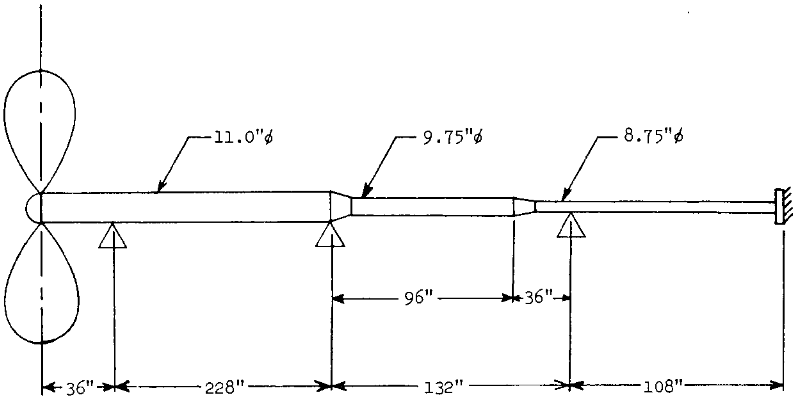

Fig. 2 Example of typical shaftline

The following data are applicable to the shaft system shown in Fig. 2:

b = 36 in. (see Fig. 1 for definition)
l = 228 in. (see Fig. 1 for definition)
d = 11 in. (for tailshaft)
$I = 0.049\, d^4 = 717.4$ in.^{4} (for d = 11.0 in.)
pg = 0.28355 lb/in.^{3} (weight density of steel)
$\mu = 0.002035\,(pg)\,(d^2)$
$= 0.07$ lb-s^2/in.^{2}
$E = 29 \times 10^6$ lb/in.^{2} (Young's modulus for steel)
I_d = 7382.8 lbs.-in.-s^2 (with 25 percent entrained water)
m = 20.72 lb-s^2/in. (with 25 percent entrained water)
N = number of propeller blades = 4

Critical speed calculation for proposed method

Calculation of influence numbers—simple support:

$$\alpha_{11} = \frac{b^3}{3EI} + \frac{b^2 l}{3EI}$$
$$= 7.47 \times 10^{-7} + 4.73 \times 10^{-6} = 5.48 \times 10^{-6}$$
$$\alpha_{12} = \frac{b^2}{2EI} + \frac{bl}{3EI}$$
$$= 3.11 \times 10^{-8} + 1.31 \times 10^{-7} = 1.62 \times 10^{-7}$$
$$\alpha_{22} = \frac{b}{EI} + \frac{l}{3EI}$$
$$= 1.73 \times 10^{-9} + 1.31 \times 10^{-7} = 5.38 \times 10^{-9}$$

Calculation of influence numbers—fixed forward end (note shortcut in arithmetic):

$$\alpha_{11} = 7.47 \times 10^{-7} + 0.75\,(4.73 \times 10^{-6}) = 4.29 \times 10^{-6}$$
$$\alpha_{12} = 3.11 \times 10^{-8} + 0.75\,(1.31 \times 10^{-7}) = 1.29 \times 10^{-7}$$
$$\alpha_{22} = 1.73 \times 10^{-9} + 0.75\,(3.65 \times 10^{-9}) = 4.47 \times 10^{-9}$$

Calculation of quantities A, B, D, and E:

$m_{\text{corrected}} = m + 1/3\,\mu\, b$
$= 20.72 + 1/3\,(0.07)\,(36) = 21.56$
$D = I_d \alpha_{22} / m_{\text{corrected}}\, \alpha_{11}$
= 0.3363 (for simple support)
= 0.3561 (for fixed forward end)
$E = (\alpha_{12})^2 / \alpha_{11}\, \alpha_{22}$
= 0.8965 (for simple support)
= 0.8766 (for fixed forward end)
$N = 4$

$$\left(1 \mp \frac{2}{N}\right) = 1/2 \text{ (for forward whirl)}$$
$$= 3/2 \text{ (for reverse whirl)}$$

$$A = \left(1 \mp \frac{2}{N}\right)(D)\,(1.0 - E)$$

= 0.0174 (forward whirl, simple support)
= 0.0522 (reverse whirl, simple support)
= 0.0220 (forward whirl, fixed forward end)
= 0.0659 (reverse whirl, fixed forward end)

$$B = \left(1 \mp \frac{2}{N}\right)(D) + 1.0$$

= 1.1681 (forward whirl, simple support)
= 1.5045 (reverse whirl, simple support)
= 1.1781 (forward whirl, fixed forward end)
= 1.5342 (reverse whirl, fixed forward end)

Calculation of ω_1^2:

$$\omega_1^2 = \frac{B \mp \sqrt{B^2 - 4A}}{2\,\alpha_{11}\, m_{\text{corrected}}\, A} \text{ (use minus sign only)}$$

Forward whirl, simple support
$\omega_1^2 = 7339$
$1/\omega_1^2 = 1.363 \times 10^{-4}$

Reverse whirl, simple support
$\omega_1^2 = 5760$
$1/\omega_1^2 = 1.736 \times 10^{-4}$

Forward whirl, fixed forward end
$\omega_1^2 = 9311$
$1/\omega_1^2 = 1.074 \times 10^{-4}$

Reverse whirl, fixed forward end
$\omega_1^2 = 7243$
$1/\omega_1^2 = 1.381 \times 10^{-4}$

Calculation of ω_2^2:

$$\omega_2^2 = \frac{KEI}{\mu l^4} = 110.0\,K$$

$\omega_2^2 = 10\,714$ for simple support (forward or reverse whirl)
$1/\omega_2^2 = 9.334 \times 10^{-5}$
$\omega_2^2 = 26\,092$ for fixed forward end (forward or reverse whirl)
$1/\omega_2^2 = 3.833 \times 10^{-5}$

Calculation of ω and limiting values of critical rpm:
Forward whirl, simple support

$$1/\omega^2 = 1.363 \times 10^{-4} + 9.334 \times 10^{-5}$$
$$= 2.296 \times 10^{-4}$$
$$\omega^2 = 4355$$

$$\omega = 66.0 \text{ rad/s} = 630.2 \text{ vpm}$$
$$\text{Critical rpm} = 157.6$$

Reverse whirl, simple support

$$1/\omega^2 = 1.736 \times 10^{-4} + 9.334 \times 10^{-5}$$
$$= 2.669 \times 10^{-4}$$
$$\omega^2 = 6862$$
$$\omega = 61.2 \text{ rad/s} = 584.5 \text{ vpm}$$
$$\text{Critical rpm} = 146.1$$

Forward whirl, fixed forward end

$$1/\omega^2 = 1.074 \times 10^{-4} + 3.833 \times 10^{-5}$$
$$= 1.457 \times 10^{-4}$$
$$\omega^2 = 6862$$
$$\omega = 82.8 \text{ rad/s} = 791.1 \text{ vpm}$$
$$\text{Critical rpm} = 197.8$$

Reverse whirl, fixed forward end

$$1/\omega^2 = 1.381 \times 10^{-4} + 3.833 \times 10^{-5}$$
$$= 1.764 \times 10^{-4}$$
$$\omega^2 = 5668$$
$$\omega = 75.3 \text{ rad/s} = 719.0 \text{ vpm}$$
$$\text{Critical rpm} = 179.7$$

Estimation of critical rpm's:
1) Forward whirl

$$\text{Critical rpm} = \frac{157.6 + 197.8}{2} = 177.7$$

Reverse whirl

$$\text{Critical rpm} = \frac{146.1 + 179.7}{2} = 162.9$$

Summary:
1. The forward whirling critical speed will occur between 157.6 and 197.8 rpm, with a most probable value of 177.7 rpm.
2. The reverse whirling critical speed will occur between 146.1 and 179.7 rpm, with a most probable value of 162.9 rpm.
3. The speed ranges from 0 to 146 rpm, and from 198 rpm up to the lowest unbalance-excited critical, will be free from serious whirling critical speeds.

Critical speed calculation by Panagopulos's formula

$$F = \frac{30}{\pi}(EI)^{1/2}[I_d(b + l/3) + mb^2(b/2 + l/3)$$
$$+ \mu(b^4/8 + lb^3/9 + 7l^4/360)]^{-1/2}$$
$$= 9.55(2.08 \times 10^{-10})^{1/2}[7382.8(112)$$
$$+ 26\,853(94) + 0.07(53\,937\,330)]^{-1/2}$$
$$= 515.9 \text{ vpm}$$

$$\text{Critical rpm} = \frac{515.9}{4} = 129 \text{ rpm}$$

It should be noted that his formula does not distinguish between forward and reverse whirl, but instead gives the lateral (not whirling) critical speed. This value normally lies between the forward and reverse whirling criticals. Notice the wide disagreement between this calculated critical rpm and those calculated by our proposed method. Also, note that the required shafting and propeller data are the same for Panagopulos's formula as for our proposed method.

Critical speed calculation using ABS computer program PROPULSOR

A total of three separate computer runs was made using different models of the shaft system. The results of these computer runs may be summarized as follows:

Run No. 1: A two-bearing model of the tailshaft and propeller was used to verify the hand calculations for a simply supported shaft. The results were as follows:

	Computer	Hand
Forward whirl rpm	170.2	157.6
Reverse whirl rpm	155.4	146.1

The PROPULSOR program does not permit the shaft end conditions to be specified, but treats all bearings as simple supports. Therefore, where a fixed forward end is required for a given run, an additional bearing, together with a short dummy shaft section, is added forward of the actual forward end bearing to provide a close approximation to the desired end condition.

Run No. 2: A dummy third bearing was added to the model of Run 1 to verify the hand calculations for a shaft with a fixed forward end. The results were as follows:

	Computer	Hand
Forward whirl rpm	201.3	197.8
Reverse whirl rpm	178.0	179.7

Run No. 3: For this run, the complete shaft line was modeled. A dummy bearing was placed at the forward end of the shaft, and an additional dummy bearing was used to simulate the fixity of the bull gear connection. This run was used to verify the "best estimated" critical speeds obtained by interpolation of the hand-calculated values. The results were as follows:

	Computer	Hand
Forward whirl rpm	191.3	177.7
Reverse whirl rpm	170.9	162.9

Conclusions

The results obtained by the hand calculations developed herein were within 7.5 percent of the computer results. This may be considered as excellent agreement, and represents a significant improvement over the results obtainable from Panagopulos's formula, which differed from the lowest of the computer results by more than twice that margin.

Appendix 3

Calculations for flexible propeller-end bearing

There are three parts to this extension of our previously derived calculation procedure:
1. Add a bearing-deflection term to the influence coefficients used in obtaining ω_1.
2. Calculate the natural frequency ω_3 of an infinitely rigid shaft on two supports, one of which is flexible.
3. Add a term to the Dunkerley equation to include the natural frequency ω_3, obtained in the preceding Part 2, in the determination of the overall system natural frequency.

In this way, all the effects of the propeller-end bearing flexibility are accounted for. The calculations will be developed only for the case of simple support at both ends of the tailshaft, since it is mathematically impossible to have lateral vibrations in a rigid shaft if the forward end of the shaft is fixed.

Referring to our model, Fig. 1, we will now place a spring of stiffness K lb/in. (or kg/mm) between the propeller-end bearing and the hull. Applying basic statics, we obtain the following corrections to the rigid-bearing influence numbers:

$$\alpha_{11} \doteq \alpha_{11}\,(\text{rigid bearing}) + \left(\frac{b+l}{Kl}\right)\left(1 + \frac{b}{l}\right)$$

$$\alpha_{12} = \alpha_{12}\,(\text{rigid bearing}) + \left(\frac{b+l}{Kl^2}\right)$$

$$\alpha_{22} = \alpha_{22}\,(\text{rigid bearing}) + \left(\frac{1}{Kl^2}\right)$$

We remember from basic vibration theory that a distributed mass, having linear deflection, may be replaced by a concentrated mass, one-third the size, acting at the point of maximum deflection. It then follows, for a shaft of mass μl, that

$$\omega_3^2 = \frac{3K}{\mu l}$$

This is the expression required for the natural frequency of a rigid shaft on one flexible bearing. In view of the simplicity of this expression, we can develop a quick means of checking the relative importance of the shaft stiffness and the bearing stiffness.

We recall from the calculations developed in the body of the paper that for a simply supported shaft on rigid bearings

$$\omega_2{}^2 = \frac{97.4EI}{\mu l^4}$$

Therefore

$$\left(\frac{\omega_3}{\omega_2}\right)^2 = \frac{Kl^3}{32.5\,EI}$$

We can go on to develop a criterion for the bearing stiffness required to limit the effect of this rigid-body motion to some arbitrary percentage reduction in the overall natural frequency. We will choose, arbitrarily, a 2 percent reduction, and apply Dunkerley's equation.

Let

$$\omega_3 = k\,\omega_2,\ \omega_{\text{overall}} = 0.98\,\omega_2$$

$$\frac{1}{\omega_2{}^2} + \frac{1}{k^2\omega_2{}^2} = \frac{1}{(0.98\omega_2)^2} \approx \frac{1.04}{\omega_2{}^2}$$

$$\frac{k^2+1}{k^2\omega_2{}^2} = \frac{1.04}{\omega_2{}^2}$$

$$k^2 + 1 = 1.04k^2$$

$$k = 5, \text{ or } \omega_3 = 5\omega_2$$

We then have

$$25 = \frac{Kl^3}{32.5EI}$$

$$K = \frac{800EI}{l^3}$$

This is the desired criterion. Note that this criterion depends only on the properties of the shaft, without considering the properties of the propeller. However, for conventional tailshaft/propeller combinations, a simple criterion of this sort should prove to be useful, at least in the early stages of design. Practical experience and engineering judgment will, of course, suggest appropriate values of the numerical constant for particular types of vessels.

Finally, we will tie all the partial system natural frequencies together, using Dunkerley's equation, to obtain the overall natural frequency ω:

$$\frac{1}{\omega^2} = \frac{1}{\omega_1{}^2} + \frac{1}{\omega_2{}^2} + \frac{1}{\omega_3{}^2}$$

where

ω_1 = natural frequency of propeller on a massless shaft, based on the modified influence numbers defined in the foregoing, found by applying the appropriate formula developed in the body of the paper

ω_2 = natural frequency of a flexible, simply supported shaft on rigid bearings, as defined in the foregoing

ω_3 = natural frequency of a rigid, simply supported shaft on one flexible and one rigid support, as defined in the foregoing

The values of the natural frequency ω obtained by this method may be used in place of those obtained for rigid, simple supports in determining the lower limits of the whirling critical, as shown in the body of the paper.

It should be noted that the development presented in this Appendix is offered without verification, for the purpose of stimulating further investigation and advancing the state of the art.